CHAPTER 2

Laplace Transform, Transfer Function and Electrical Systems Modeling

By

Dr. Ayman Yousef

Modified By

Dr. Emad Sami

Transfer Function

Transfer Function (TF)

* Is a mathematical model that defines the relationship between the output and the input of a linear system. It is usually expressed as the ratio of the Laplace transform of the output function divided by the Laplace transform of the input function.

* The order of the numerator must not exceed that of the denominator.

$$G(s) = \frac{Y(s)}{R(s)}$$

Transfer function

- A transfer function is defined by $G(s)$

$$G(s) = \frac{Y(s)}{R(s)}$$

Laplace transform of system output

Laplace transform of system input

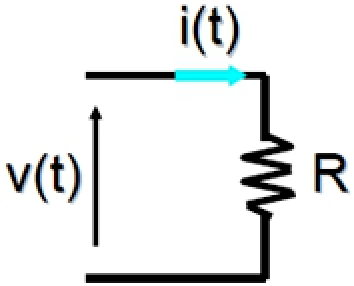
$$Y(s) = G(s)R(s)$$

- A system is assumed to be at rest. (Zero initial condition)

Electrical Systems Modeling

Models of electrical elements

Resistance

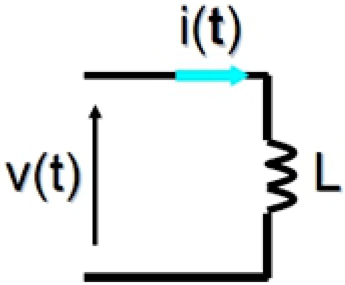


$$v(t) = Ri(t)$$

Laplace transform

$$\frac{V(s)}{I(s)} = R$$

Inductance

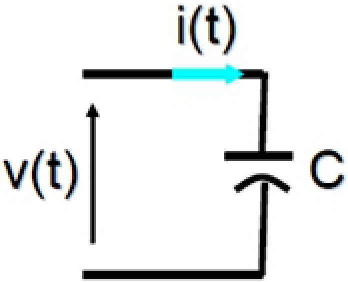


$$v(t) = L\frac{di(t)}{dt}$$

$$(i(0) = 0)$$

$$\frac{V(s)}{I(s)} = sL$$

Capacitance



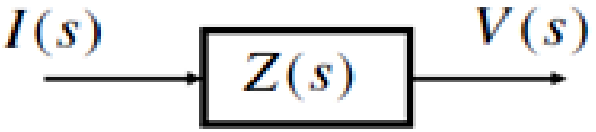
$$i(t) = C\frac{dv(t)}{dt}$$

$$(v(0) = 0)$$

$$\frac{V(s)}{I(s)} = \frac{1}{sC}$$

Impedance Modeling

In s - domain, $V(s) = Z(s)\, I(s)$ $(Z(s)$ - effective resistance)

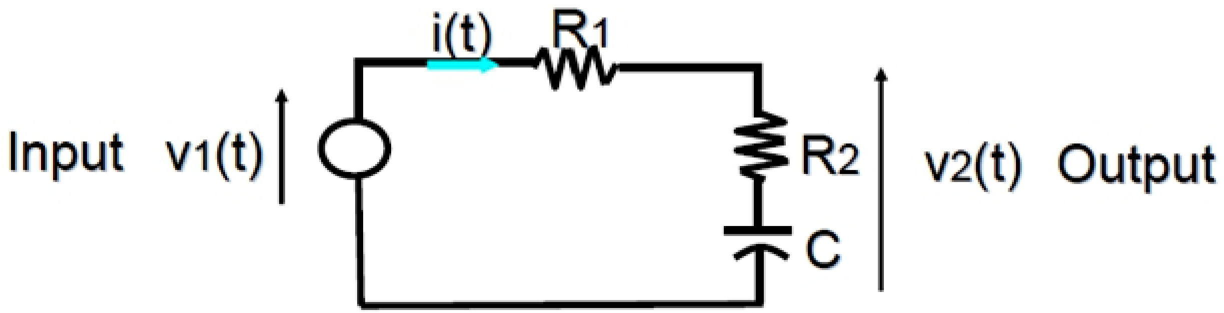


Element	t – domain	s – domain	Impedance
Resistance	$v(t) = Ri(t)$	$V(s) = RI(s)$	R
Inductor	$v(t) = L\frac{di(t)}{dt}$	$V(s) = sL\, I(s)$	sL
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$V(s) = \frac{1}{sC} I(s)$	$\frac{1}{sC}$

Impedance in series: $Z_{eff}(s) = Z_1(s) + Z_2(s) + \cdots$

Impedance in paralell: $Z_{eff}(s) = \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)} + \cdots$

Modeling example



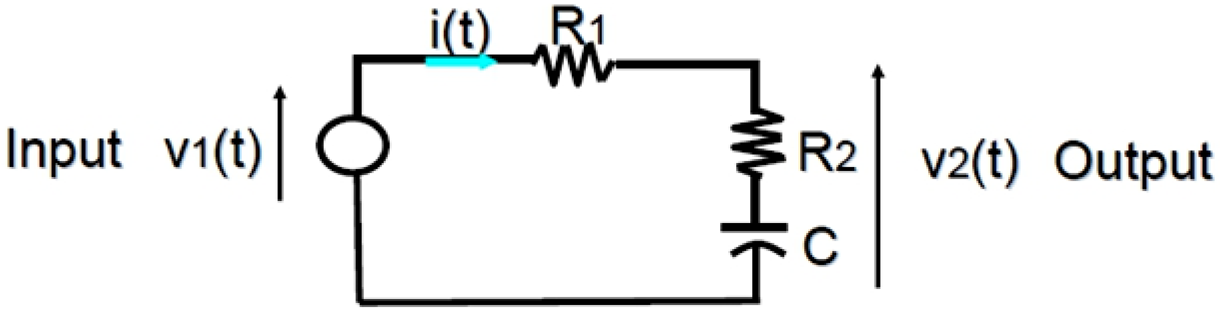
- Kirchhoff voltage law (with zero initial conditions)

$$\begin{aligned} v_1(t) &= (R_1 + R_2)i(t) + \tfrac{1}{C}\textstyle\int_0^t i(\tau)d\tau \\ v_2(t) &= R_2 i(t) + \tfrac{1}{C}\textstyle\int_0^t i(\tau)d\tau \end{aligned}$$

- By Laplace transform,

$$\begin{aligned} V_1(s) &= (R_1 + R_2)I(s) + \tfrac{1}{sC}I(s) \\ V_2(s) &= R_2 I(s) + \tfrac{1}{sC}I(s) \end{aligned}$$

Modeling example (cont'd)



- Transfer function

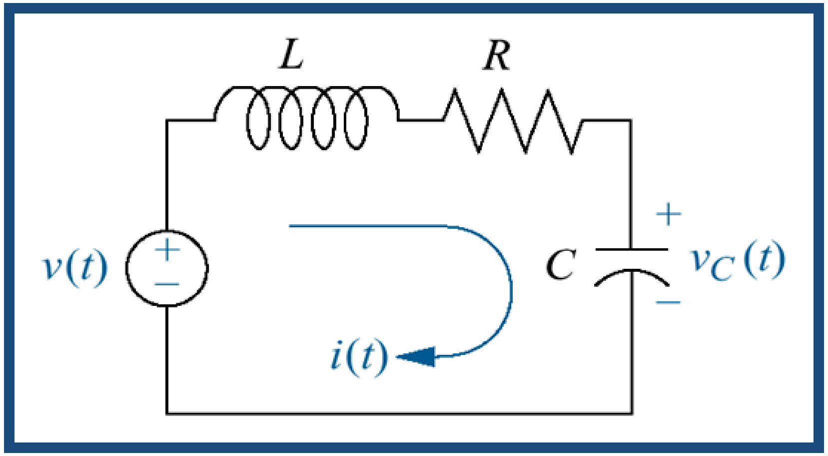
$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{R_2 + \frac{1}{sC}}{(R_1+R_2)+\frac{1}{sC}}$$

$$= \frac{R_2 C s + 1}{(R_1+R_2)Cs+1}$$

(first-order system)

Example 2

Find the transfer function relating the v_c(t) to the input voltage v(t).

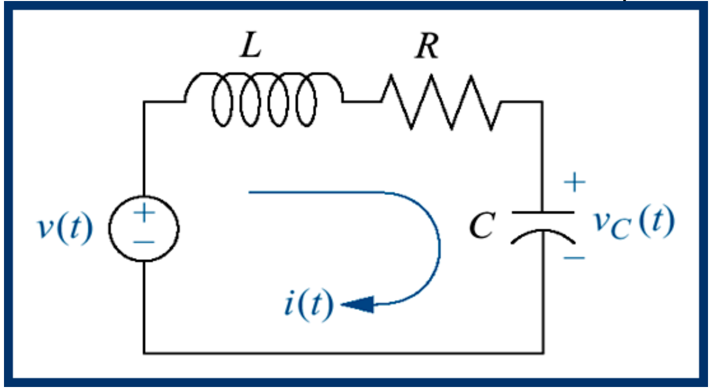


Summing the voltages around the loop, assuming zero initial conditions, yields the integro-differential equation for this network as

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$

Example 2 (Cont'd)

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$



$$i(t) = c\frac{dv_c}{dt}$$

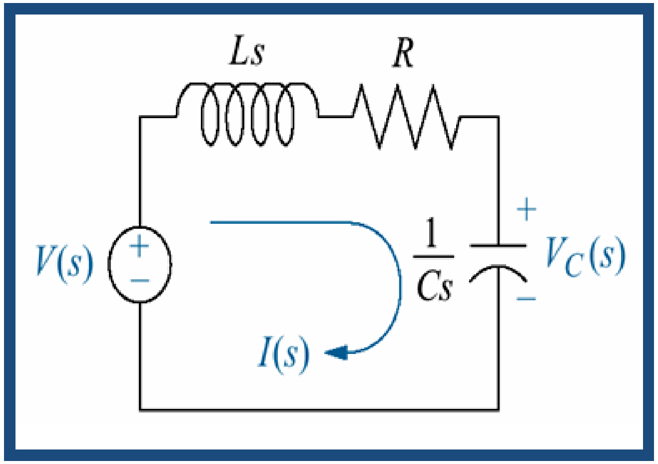
Taking Laplace $I(s) = csv_c(s)$ substitute in above eq.

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \quad \Longrightarrow \quad \frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

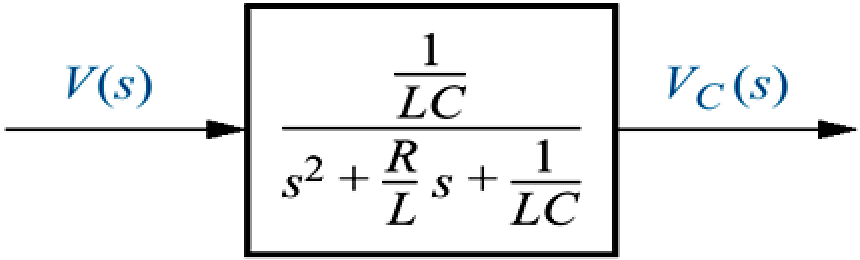
Example 3

Repeat Example 2 using the transformed circuit.

Solution



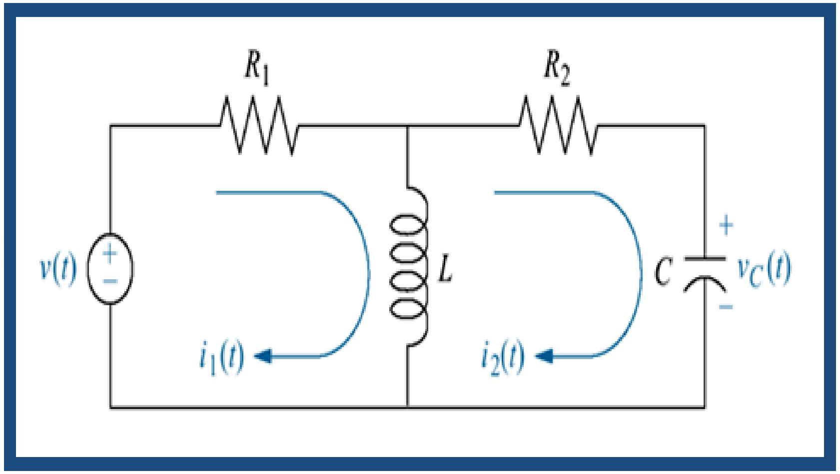
using voltage division



$$\frac{\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

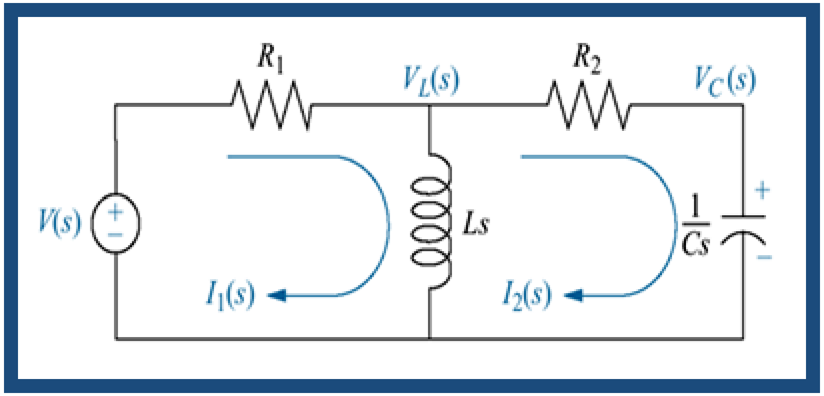
Example 4

Find the T.F $I_2(s)/V(s)$



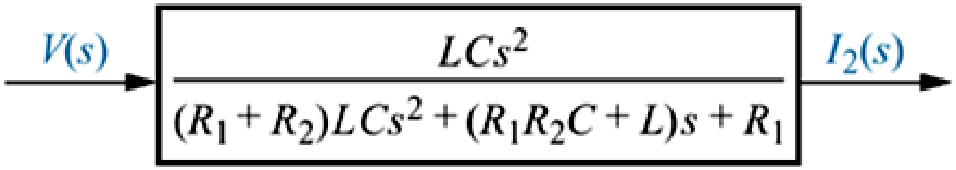
Solution

Using mesh current



$$(R_1 + LS)I_1 - LSI_2 = V(s)$$

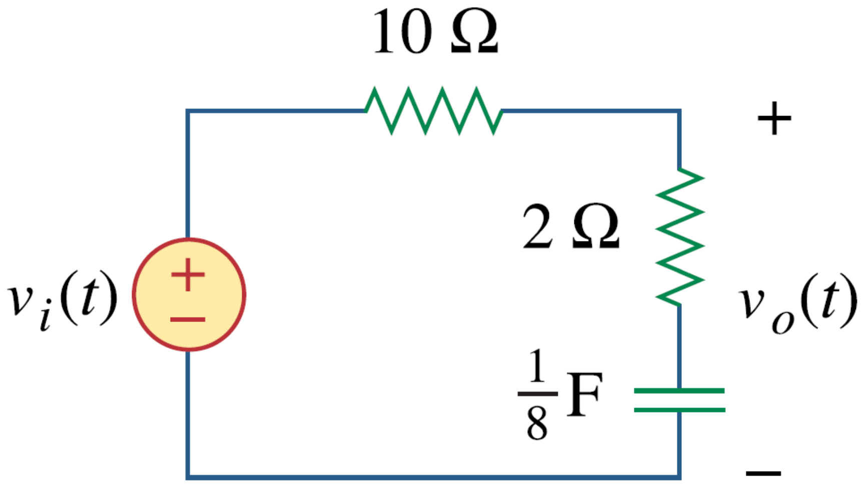
$$-LSI_1 + (R_2 + LS + 1/CS)I_2 = 0$$



$$\frac{LCs^2}{(R_1+R_2)LCs^2+(R_1R_2C+L)s+R_1}$$

Example 5

Obtain the transfer function $\mathbf{V}_o(s)/\mathbf{V}_i(s)$



$$V_i(s) = (10 + 2 + (1/(\frac{s}{8})) * I(s)$$

$$V_o(s) = (2 + (1/(\frac{s}{8})) * I(s)$$

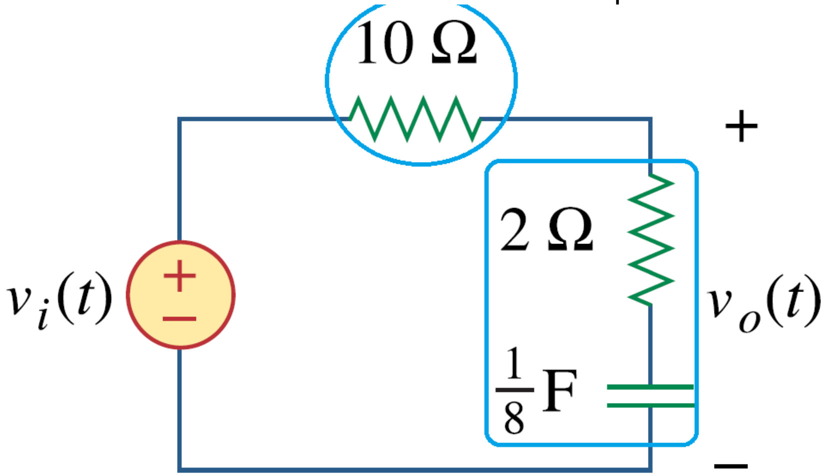
$$\frac{V_o(s)}{V_i(s)} = \frac{(2 + (1/(\frac{s}{8}))}{(10 + 2 + (1/(\frac{s}{8}))} = \frac{s+4}{6s+4}$$

Example 5: continued

Obtain the transfer function $\mathbf{V}_o(s)/\mathbf{V}_i(s)$

Using:

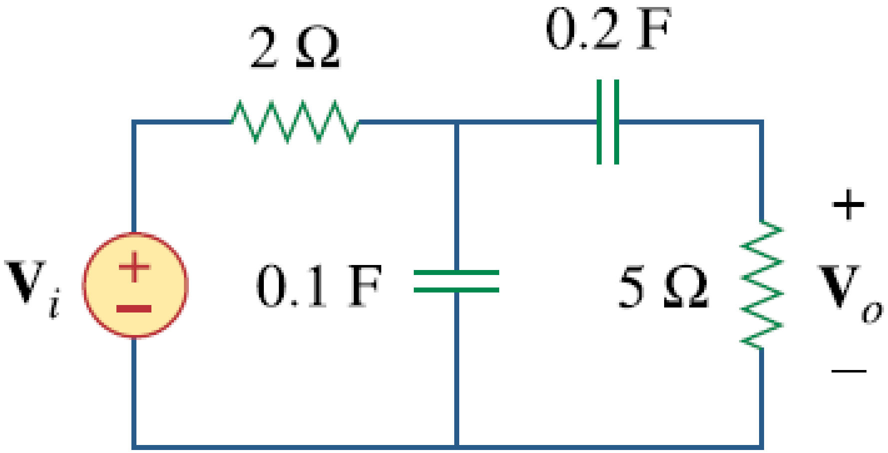
Voltage Divider Rule



$$\frac{V_o(s)}{V_i(s)}=\frac{\boxed{(2+(1/(\frac{s}{8}))}}{(\boxed{10}+\boxed{2+(1/(\frac{s}{8}))})}=\frac{s+4}{6s+4}$$

Example 6

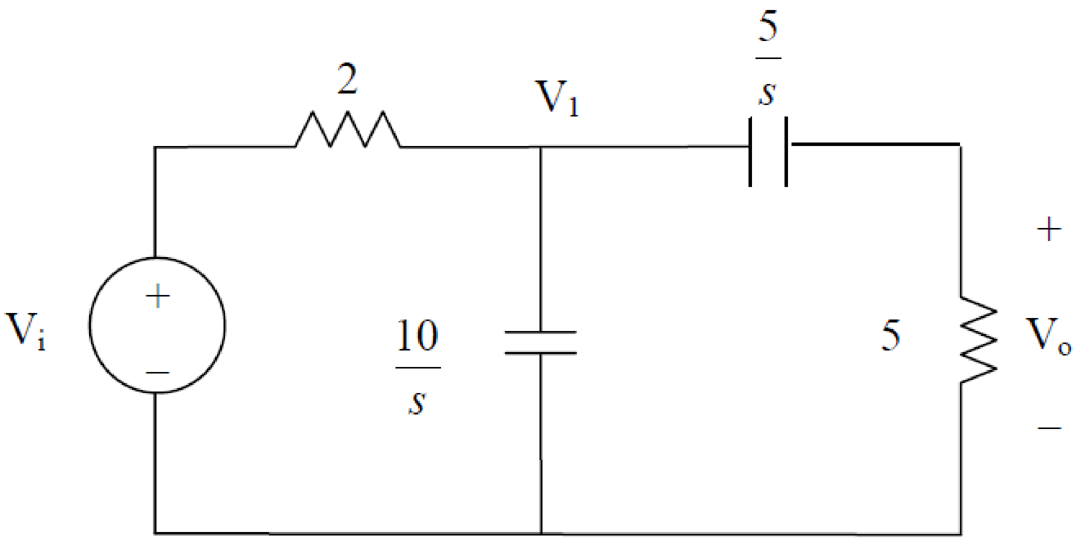
Obtain the transfer function $\mathbf{V}_o(s)/\mathbf{V}_i(s)$



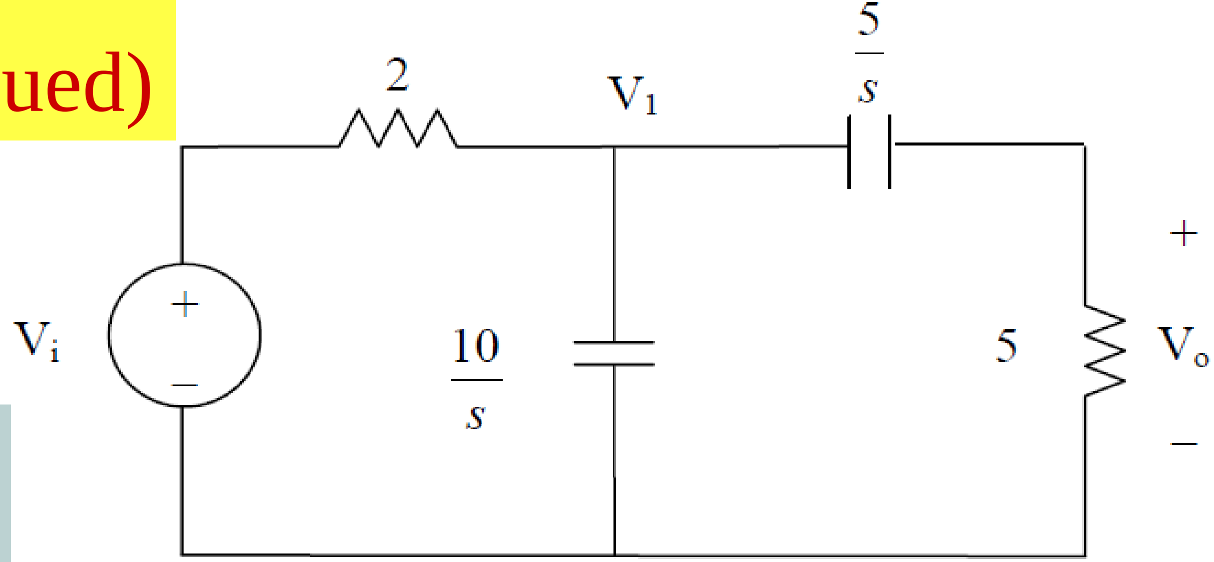
Solution

$$0.2F \longrightarrow \frac{1}{CS}=\frac{1}{(0.2)S}=\frac{5}{S}$$

$$0.1F \longrightarrow \frac{1}{(0.1)S}=\frac{10}{S}$$



Example 6: (Continued)



Using:

Voltage Divider Rule

$$Z=\frac{10}{s}//(5+\frac{5}{s})=\frac{\frac{10}{s}(5+\frac{5}{s})}{5+\frac{15}{s}}=\frac{\frac{10}{s}5(\frac{1+s}{s})}{\frac{5}{s}(3+s)}=\frac{10(s+1)}{s(s+3)}$$

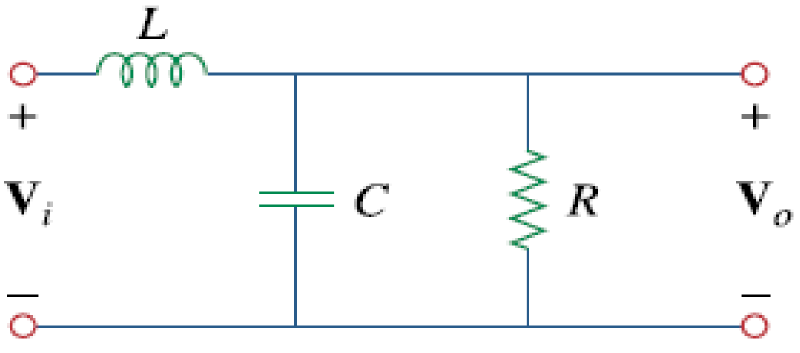
$$\frac{V_1}{V_i}=\frac{Z}{Z+2} \Longrightarrow V_i=\frac{Z+2}{Z}V_1$$

$$\frac{V_o}{V_1}=\frac{5}{5+5/s} \Longrightarrow V_o=\frac{5}{5+5/s}V_1$$

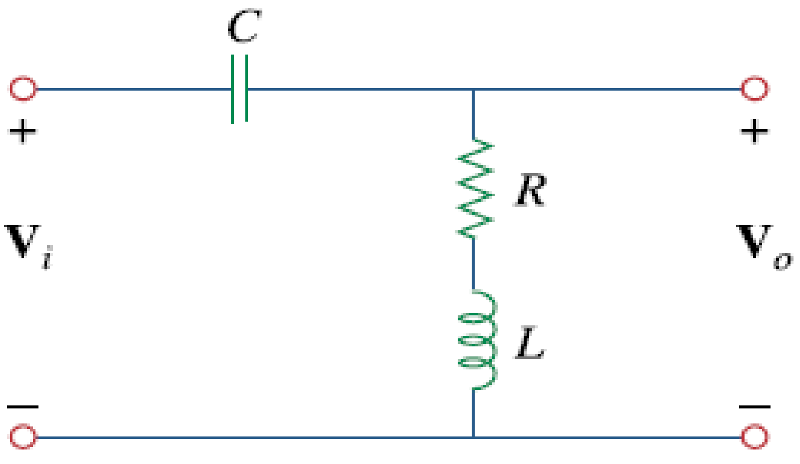
$$\boxed{\frac{V_o}{V_i}=\frac{5s}{s^2+8s+5}}$$

Example 7

Obtain the transfer function $\mathbf{V}_o(s)/\mathbf{V}_i(s)$



(a)



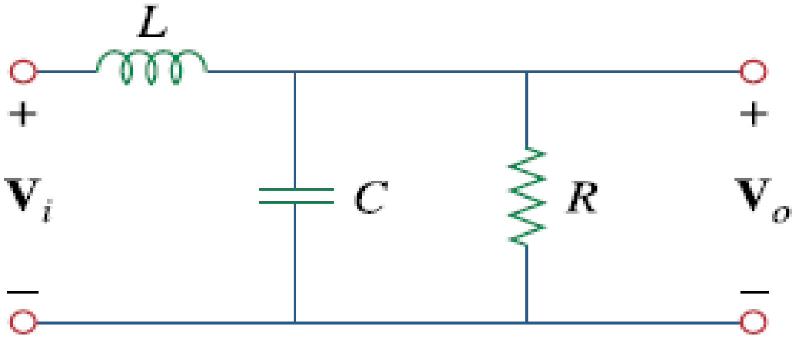
(b)

Example 7

Obtain the transfer function $\mathbf{V}_o(s)/\mathbf{V}_i(s)$

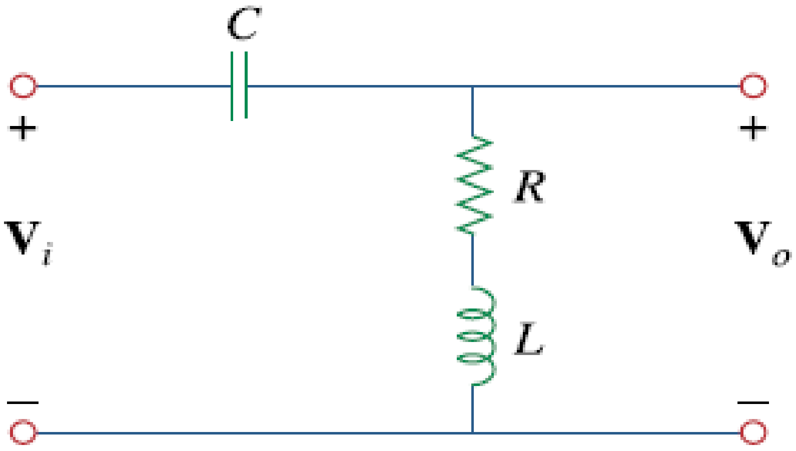
Solution

$$\frac{V_o(s)}{V_i(s)} = \frac{R}{RLC\,S^2 + L\,S + R}$$



(a)

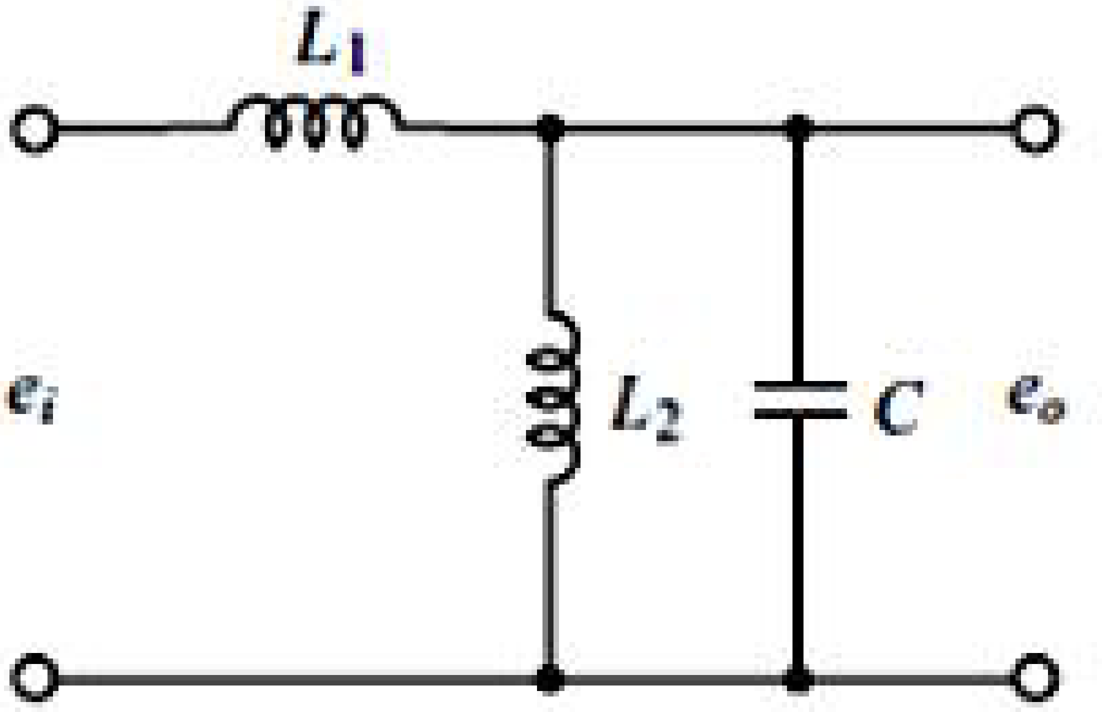
$$\frac{V_o(s)}{V_i(s)} = \frac{LC\,S^2 + RCS}{LC\,S^2 + RCS + 1}$$



(b)

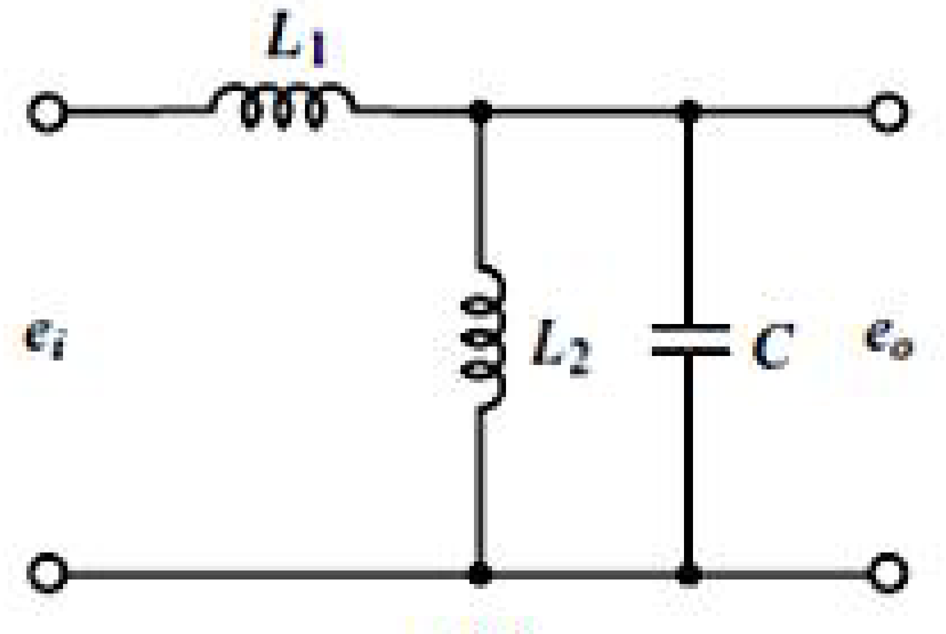
Example 8

For the following circuits e_i and e_o are in time domain, Determine the transfer function $E_o(s)/E_i(s)$ of each circuit and draw its block diagram.



Example 8: (continued)

Solution



$$\frac{E_o(s)}{E_i(s)}=\frac{\boxed{L_2C}S^2}{\boxed{L_1L_2C}S^3+\boxed{L_2C}S^2+\boxed{L_1}S}$$

Summary

- Introduction:

– Laplace transform

– Transfer function

– Modeling of electrical systems

Next Lecture:

Block diagrams and Signal flow graphs